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Satoshi Kobayashi , Shinji Ogihara & Nobuo Takeda

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# Damage mechanics analysis for predicting mechanical behavior of general composite laminates containing transverse cracks

SATOSHI KOBAYASHI [1,*], SHINJI OGIHARA [2] and NOBUO TAKEDA [3]

[1] *Takeda Laboratory, Komaba Open Laboratory, The University of Tokyo, 4-6-1 Komaba, Meguro-ku, Tokyo 153-8904, Japan*
[2] *Department of Mechanical Engineering, Science University of Tokyo, 2641 Yamasaki, Noda, Chiba 278-8510, Japan*
[3] *Graduate School of Frontier Sciences, The University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo 113-8656, Japan*

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**Abstract**—A damage mechanics analysis is applied to predict the relation between laminate stress and strain in composite laminates containing transverse matrix cracks. First, the transverse crack density as a function of laminate stress is predicted based on both energy and average stress criteria. Next, the laminate strain increment caused by transverse cracking is also derived as a function of applied laminate stress and transverse crack density, and then the relation between laminate stress and strain is predicted considering matrix cracking. The analytical prediction of the relation between laminate stress and strain is compared with the experimental results. The analytical prediction slightly underestimates the laminate stress because the strain increment associated with transverse crack opening displacement is overestimated in this analysis. An advantage of the present method is that it can be applied to laminates with arbitrary stacking sequences.

*Keywords*: Composite laminate; transverse crack; energy release rate; damage mechanics; crack opening displacement.

## 1. INTRODUCTION

In the failure process of fiber reinforced composite laminates, it is well known that many types of damages occur, that is, fiber–matrix interfacial debonding, transverse cracks, delamination and fiber breaks. Among them, transverse cracks are the first significant mode of damage observed in the weakest laminae. Transverse

*To whom all correspondence should be addressed. E-mail: psycho@compmat.rcast.u-tokyo.ac.jp

cracks cause the strain increments which result in nonlinear behavior in the relation between laminate stress and strain.

Many studies have been conducted on transverse cracking in laminated composites. Shear-lag analysis [1–4], variational stress analysis [5–7] and approximate elastic analysis [8, 9] were conducted for cross-ply laminates. Moreover, there are few studies to predict the nonlinear relation between laminate stress and strain caused by transverse cracking [10] which is necessary for damage tolerance design.

Recently, Gudmundson and Zang proposed a damage mechanics analysis for the prediction of thermoelastic properties of composite laminates containing matrix cracks [11]. In damage mechanics analysis, generally speaking, parameters relating internal damage state and material properties must be obtained by using a numerical calculation or an experiment. Gudmundson and Zang assumed that the average crack opening displacements for a row of transverse cracks in a ply is approximated by the average crack opening displacements for the same row of cracks in an infinite homogeneous transversely isotropic medium which has the same properties as the ply in consideration, and derived the relation between thermoelastic properties and transverse crack density successfully. This analysis can be applied to general laminates.

In this study, the energy release rate associated with transverse cracking and the average stress of the plies containing transverse cracks are derived using damage mechanics analysis [11]. Transverse cracking behavior in 90° plies in a composite laminate is predicted based on both the energy and average stress criteria [9]. In the damage mechanics analysis, strain increments associated with transverse cracking can be derived. The global laminate strain is represented as the sum of the undamaged laminate strain and the laminate strain increment. An advantage of this method is that it can be applied to general laminates with arbitrary stacking sequence.

## 2. ANALYSIS

### *2.1. Damage mechanics analysis for prediction of thermoelastic properties of laminates containing transverse cracks*

Gudmundson and Zang [11] proposed a model for the prediction of thermoelastic properties of composite laminates containing transverse cracks whose surfaces are parallel to the fiber direction and perpendicular to the laminate plane. In the present study, in-plane loading is solely considered. The laminate which consists of $N$ plies with transverse cracks is considered as shown in Fig. 1. The normalized transverse crack density for ply $k$, $\rho^k$ is defined as

$$\rho^k = \frac{a^k}{d^k}, \tag{1}$$

where $a^k$ is thickness of ply $k$, $d^k$ is the average distance of transverse cracks.

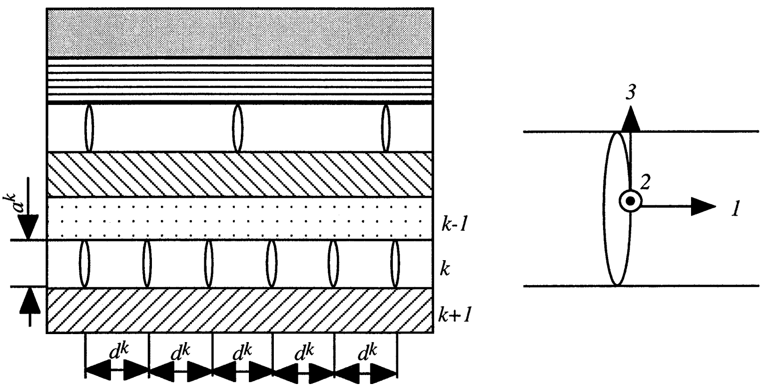


**Figure 1.** General laminates with transverse cracks.

Using the classical lamination theory, the tractions on prospective crack surface, $\boldsymbol{\tau}^k$, which is related to the average crack opening displacement, is represented as

$$\boldsymbol{\tau}^k = \mathbf{A}^k \boldsymbol{\varepsilon} + (\mathbf{C}^k - \mathbf{A}^k \boldsymbol{\alpha}_0)\Delta T, \tag{2}$$

where

$$\begin{aligned} \mathbf{A}^k &= \mathbf{N}^k (\mathbf{S}^k)^{-1}, \\ \mathbf{C}^k &= \mathbf{A}^k (\boldsymbol{\alpha}_0 - \boldsymbol{\alpha}^k), \end{aligned} \tag{3}$$

and $\boldsymbol{\varepsilon}$ is the applied laminate strains, $\boldsymbol{\alpha}_0$ is the in-plane thermal expansion coefficient of the undamaged laminate and $\boldsymbol{\alpha}^k$ is the in-plane thermal expansion coefficient of ply $k$. The matrix $\mathbf{N}^k$ in equation (3) is represented by the unit normal vector $n_j^k$ on the crack surface in ply $k$, as shown in the Appendix 1. $\mathbf{S}^k$ is the in-plane compliance of ply $k$. $\Delta T$ is temperature change from the stress free temperature, $T_{\mathrm{sf}}$. In equation (2), the residual stresses and strains due to other reasons than the mismatch in thermal expansion coefficients are neglected, while they appear in Ref. [11].

The average transverse crack opening displacements in ply $k$, $\Delta\mathbf{u}^k$, linearly depend on all crack surface tractions as

$$\Delta\mathbf{u}^k = a^k \sum_{i=1}^{N} \boldsymbol{\beta}^{ki} \boldsymbol{\tau}^i, \tag{4}$$

where $\boldsymbol{\beta}^{ki}$ $(k, i = 1, 2, \ldots, N)$ are $3 \times 3$ matrices which depend on laminate lay-ups, ply properties and transverse crack densities, and shown in the Appendix 1. According to equations (2)–(4), the average increment strain, $\Delta\boldsymbol{\varepsilon}^k$, due to crack opening displacement in ply $k$ can be written as

$$\Delta\boldsymbol{\varepsilon}^k = \left(\rho^k / a^k\right)\left(\mathbf{N}^k\right)^T \Delta\mathbf{u}^k$$

$$= \rho^k \left(\mathbf{N}^k\right)^T \sum_{i=1}^{N} \boldsymbol{\beta}^{ki} [\mathbf{A}^i \boldsymbol{\varepsilon} + (\mathbf{C}^i - \mathbf{A}^i \boldsymbol{\alpha}_0) \Delta T]. \tag{5}$$

In a cracked ply $k$, there is a distinction between the effective strains, $\boldsymbol{\varepsilon}^{k(e)}$, and average strains, $\boldsymbol{\varepsilon}^{k(a)}$. The effective strains are the strains measured on a global scale, while the average strains are the strains which are experienced by the material. The relation between them can be written as

$$\boldsymbol{\varepsilon}^{k(e)} = \boldsymbol{\varepsilon}^{k(a)} + \Delta \boldsymbol{\varepsilon}^k. \tag{6}$$

Under in-plane loading conditions, the laminate effective in-plane strains, $\boldsymbol{\varepsilon}^*$ and effective strains of ply $k$ are equal, that is,

$$\boldsymbol{\varepsilon}^* = \boldsymbol{\varepsilon}^{k(e)}. \tag{7}$$

By the aid of the classical lamination theory, the relations between stresses and strains for a laminate containing transverse cracks is expressed as

$$\begin{aligned} \boldsymbol{\varepsilon}^* &= \mathbf{S} \boldsymbol{\sigma} + \boldsymbol{\alpha} \Delta T, \\ \boldsymbol{\varepsilon}^{k(a)} &= \mathbf{S}^k \boldsymbol{\sigma}^{k(a)} + \boldsymbol{\alpha}^k \Delta T, \end{aligned} \tag{8}$$

where $\boldsymbol{\sigma}$ is laminate in-plane stresses, $\mathbf{S}$ and $\boldsymbol{\alpha}$ are in-plane compliance matrix and in-plane thermal expansion coefficients of a laminate with transverse cracks and can be expressed as

$$\mathbf{S} = \left( (\mathbf{S}_0^{-1}) - \sum_{k=1}^{N} \nu^k \rho^k (\mathbf{A}^k)^T \sum_{i=1}^{N} \boldsymbol{\beta}^{ki} \mathbf{A}^i \right)^{-1}, \tag{9}$$

$$\boldsymbol{\alpha} = \boldsymbol{\alpha}_0 + \mathbf{S} \sum_{k=1}^{N} \nu^k \rho^k (\mathbf{A}^k)^T \sum_{i=1}^{N} \boldsymbol{\beta}^{ki} \mathbf{C}^i, \tag{10}$$

where $\mathbf{S}_0$ is the in-plane compliance matrix of the undamaged laminate and $\nu^k$ is the volume fraction of ply $k$.

### *2.2. Derivation of energy release rate associated with transverse cracking [12]*

The inverse of (1, 1) component of in-plane compliance matrix $\mathbf{S}$ is regarded as the laminate Young's modulus. Therefore, the laminate Young's modulus, $E(\rho^k)$ with transverse cracks in ply $k$ whose density is $\rho^k$, can be expressed as

$$E_{\mathrm{LAM}}(\rho^k) = \frac{1}{\mathbf{S}_{(1,1)}}. \tag{11}$$

Here, only the transverse cracking in 90° ply under uniaxial tension is considered. In the present analysis, material properties are assumed to be independent of temperature. When the normalized transverse crack density becomes $\rho^k$ from $\rho^k/2$ under a constant laminate tensile stress, $\sigma$, the potential energy per unit volume

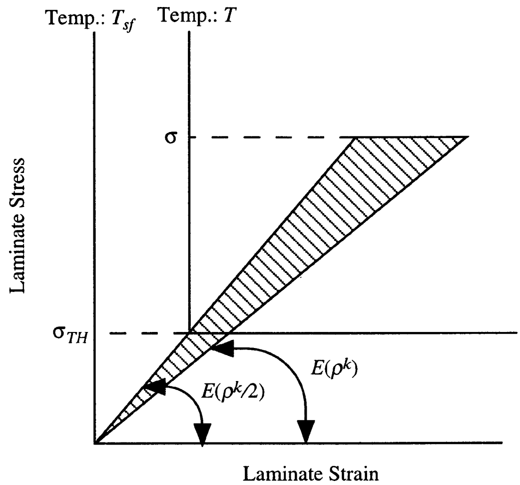


**Figure 2.** Schematic of stress–strain curve before and after new transverse cracking between two existing transverse cracks.

released by transverse cracking is expressed as the shaded area in the Fig. 2. Considering the variation of the potential energy of the laminate length $d^k$, the energy release rate associated with transverse cracking, $G(\rho^k)$, is expressed using the laminate Young's modulus as a function of transverse crack density as

$$G(\rho^k) = \frac{\sum_{i=1}^{N} a^i}{\rho^k}(\sigma - \sigma_{\mathrm{TH}})^2\left(\frac{1}{E_{\mathrm{LAM}}(\rho^k)} - \frac{1}{E_{\mathrm{LAM}}(\rho^k/2)}\right), \tag{12}$$

where $\sigma_{\mathrm{TH}}$ is a parameter to consider the effect of thermal residual stresses due to cure process. Considering the 90° ply transverse cracking, $\sigma_{\mathrm{TH}}$ is an axial laminate stress to vanish the thermal residual stress in 90° ply in loading direction, that is, $\sigma_{\mathrm{TH}}$ is the stress required to close transverse cracks as shown in the Appendix 2.

### *2.3. Prediction of transverse cracking*

To predict transverse cracking, both the stress criterion [1] and the energy criterion [2, 3, 6, 9] have been used. The energy criterion is shown to be very effective when the cracking ply is very thin and the stiffness of the neighboring plies is large. In the present study, both the energy and stress criteria are considered [9].

In the energy criterion, it is assumed that transverse cracks onset when the energy release rate, $G(\rho^k)$, reaches a critical value, $G_c$. By substituting $G_c$ into $G$ in equation (12), the laminate stress when the normalized transverse crack density reaches $\rho^k$, is expressed as

$$\sigma(\rho^k) = \sqrt{\frac{G_c \sum_{i=1}^{N} a^i}{\rho^k}\left(\frac{1}{E_{\mathrm{LAM}}(\rho^k)} - \frac{1}{E_{\mathrm{LAM}}(\rho^{k/2})}\right)^{-1}} + \sigma_{\mathrm{TH}}. \tag{13}$$

In the stress criterion, it is usually assumed that a crack occurs when the maximum tensile stress reaches the ply strength. Unfortunately, the stress distribution is not obtained by the present damage mechanics analysis. Instead, only the average stresses in the plies are available. Therefore we take the average stress as a measure of transverse cracking. We call this concept the average stress criterion.

According to equation (8), the average in-plane stress in cracked ply $k$, $\sigma^{k(a)}$ is

$$\boldsymbol{\sigma}^{k(a)} = \left(\mathbf{S}^k\right)^{-1}\left(\boldsymbol{\varepsilon}^{k(a)} - \boldsymbol{\alpha}^k \Delta T\right). \tag{14}$$

For the case of uniaxial tension, average tensile stress in ply $k$ (90° ply), $\sigma_1^{k(a)}$ is expressed as

$$\begin{aligned}\sigma_1^{k(a)} = &\left(1 - \rho^k \beta_1^k Q_{11}^k\right)\left[\left\{Q_{11}^k S_{11} + Q_{12}^k S_{12}\right\}\sigma\right. \\ &\left. -\left\{\left(\alpha_1^k - \alpha_1\right)Q_{11}^k + \left(\alpha_2^k - \alpha_2\right)Q_{12}^k\right\}\Delta T\right],\end{aligned} \tag{15}$$

where $\beta_i$, $Q_{ij}^k$, $S_{ij}$, $\alpha_i^k$, $\alpha_i$ are the components of $\boldsymbol{\beta}$, $(\mathbf{S}^k)^{-1}$, $\mathbf{S}$, $\boldsymbol{\alpha}^k$, $\boldsymbol{\alpha}$, respectively. In the average stress criterion, it is assumed that the transverse cracks onset when the average stress normal to crack surfaces for the ply $k$ reaches a critical value, $\sigma_c^k$.

$$\sigma(\rho^k) = \frac{\dfrac{\sigma_c^k}{(1 - \rho^k \beta_1^k Q_{11}^k)} + \{(\alpha_1^k - \alpha_1)Q_{11}^k + (\alpha_2^k - \alpha_2)Q_{12}^k\}\Delta T}{Q_{11}^k S_{11} + Q_{12}^k S_{12}}. \tag{16}$$

Figure 3 shows the numerical examples for transverse cracking in 90° plies of carbon fiber /bismaleimide composite G40-800/5260 $[0/90/\pm45]_s$ laminates with one 90° ply located symmetrical to the center plane and $[\pm45/0/90]_s$ laminates with two 90° plies at the center. These laminates has the same Young's modulus so that the effect of the stacking sequence (thickness of 90° plies) may be clarified. Material properties used are shown in Table 1. It should be noted that transverse crack density is different form the normalized transverse crack density mentioned above.

The predictions based on both energy and average stress criteria are shown in Fig. 3. Though analytical predictions based on the average stress criterion show

**Table 1.**
Material properties of G40-800/5260 unidirectional composite used in the analysis

| | |
|---|---|
| Longitudinal Young's modulus (GPa) | 152 |
| Transverse Young's modulus (GPa) | 10.0 |
| In-plane shear modulus (GPa) | 6.94 |
| Longitudinal Poisson's ratio | 0.33 |
| Transverse Poisson's ratio | 0.49 |
| Longitudinal thermal expansion coefficient ($10^{-6}$/°C) | −0.6 |
| Transverse thermal expansion coefficient ($10^{-6}$/°C) | 36.0 |
| Stress free temperature (°C) | 195 |

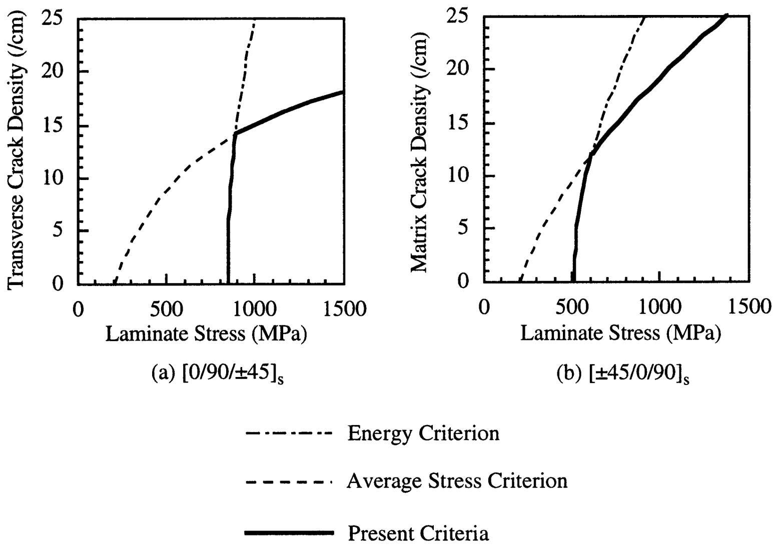


**Figure 3.** Analytical prediction of transverse crack density in 90° ply of $[0/90/\pm 45]_s$ and $[\pm 45/0/90]_s$ laminate as a function of laminate stress based on both energy and average stress criteria ($G_c = 330\ \mathrm{J/m^2}$, $\sigma_c^k = 75$ MPa).

little effect of the stacking sequence, predictions based on the energy criterion show the large effect of ply thickness. That is, transverse crack onset stress becomes larger and transverse crack density growth rate becomes smaller as the ply thickness becomes thinner, when the energy criterion is applied.

The authors have shown that we have to regard the criterion that gives lower transverse crack density at the same laminate strain as the prediction when considering both the energy and stress criteria [9]. This concept is based on the idea that a transverse crack occurs when both the energy and stress criteria are satisfied. In the present case, the proposed analytical prediction is expressed by the bold line in Fig. 3.

### *2.4. Prediction of the relation between laminate stress and strain*

In the previous section, the relation between normalized transverse crack density and laminate stress is derived. Here, the laminate strain increment caused by transverse cracking is considered. It is assumed that transverse cracks of amount $\rho^k$ initiate at a constant laminate stress in the laminate without transverse cracks, as shown Fig. 4. The strain increment of laminate strain can be calculated with equation (5), that is, total laminate strain is expressed as equation (7). Total laminate strain after transverse cracking also can be calculated considering unloading process as

$$\varepsilon^* = \frac{\sigma(\rho^k)}{E_{\mathrm{LAM}}(0)} + \Delta\varepsilon(\rho^k, \boldsymbol{\sigma})$$

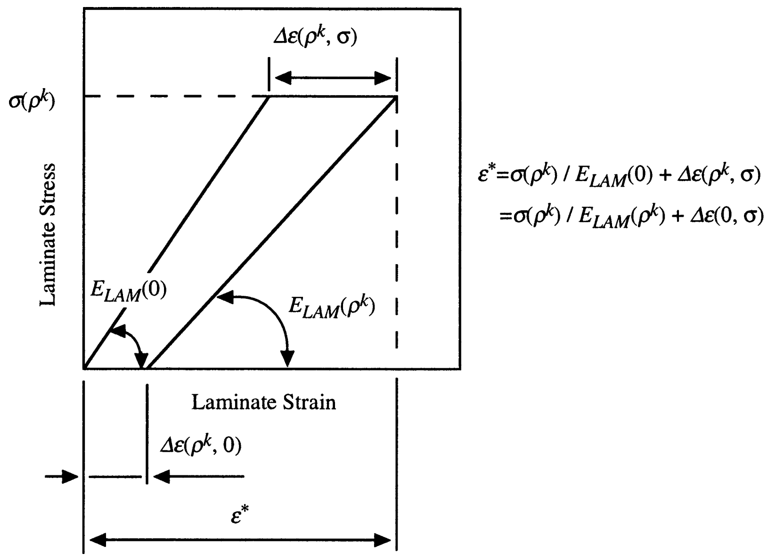


**Figure 4.** Schematic of stress−strain curve before and after transverse cracking.

$$= \frac{\sigma(\rho^k)}{E_{\mathrm{LAM}}(\rho^k)} + \Delta\varepsilon(\rho^k, \mathbf{0}). \tag{17}$$

## 3. DISCUSSION

In order to verify the efficiency of the present analysis, the prediction of the transverse crack behavior by the present analysis are compared to the experimental results obtained by the authors. The material system used in the experiments is bismaleimide-based CFRP, G40-800/5260. Laminate configurations tested are $[0/90]_s$, $[\pm 45/90]_s$, $[\pm 45/0/90]_s$ and $[0/90/\pm 45]_s$. The specimen size was 150 mm long and 25 mm wide. GFRP tabs were glued on the specimens which resulted in a specimen gage length of 90 mm. Quasi-static tensile tests were conducted at room temperature (25°C). The crosshead speed was 0.5 mm/min. During the test, the testing machine was periodically stopped, and the specimen was observed by a soft X-ray radiography. The number of transverse cracks in 90° ply was counted to obtain the transverse crack density, which was defined as the number of transverse cracks per unit length. Transverse crack multiplication in 90° plies, which spanned the laminate width, were observed in all laminate configuration. Matrix cracks in ±45° plies were also observed in $[\pm 45/90\ ]_s$ laminates; however, the growth of these cracks in the width direction is very small. That is, matrix cracks in ±45° plies are ineffective in the macroscopic mechanical behavior of laminates.

Figure 5 shows the predictions and the experimental results of transverse crack behavior in 90° plies. The predictions which satisfy both energy and average stress criteria are shown. Material properties used in the analysis are in Table 1. In the present analysis, critical values are necessary for the prediction. In this paper,

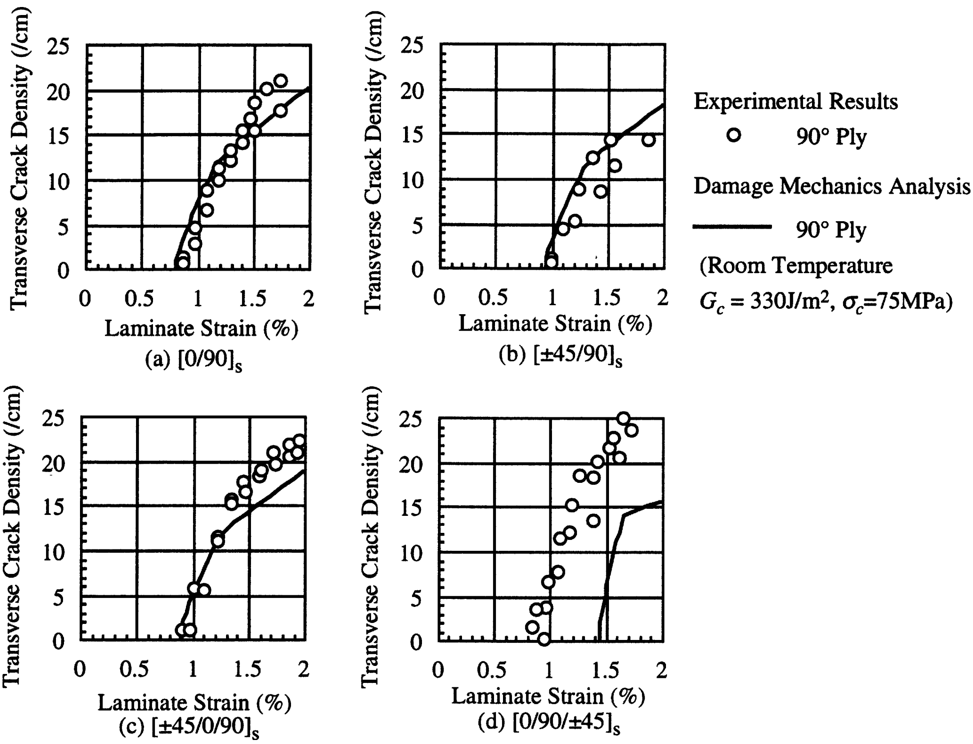


**Figure 5.** Transverse crack density in 90° plies as a function of laminate strain (experimental results and analytical prediction).



the critical energy release rate, $G_c$ was determined by the stress which gives the first cracking in $[0/90]_s$ laminate, 330 J/m$^2$ and the critical average stress, $\sigma_c$, was assumed to be the transverse tensile strength of the unidirectional composite, 75 MPa. For the laminate configurations mentioned above, predictions based on the energy criterion give the analytical predictions which satisfy the both criteria in the early stage of transverse cracking, and predictions based on the average stress criterion satisfy both criteria in the later stage.

Analytical predictions show good agreement with the experimental results for the $[0/90]_s$, $[\pm 45/90]_s$ and $[\pm 45/0/90]_s$ laminates. However, prediction for the $[0/90/\pm 45]_s$ laminate underestimates the transverse crack density. The critical energy release rate selected (330 J/m$^2$) provides high transverse crack onset strain for the $[0/90/\pm 45]_s$ laminate. This means that the critical energy release rate might be smaller than 330 J/m$^2$, and then predictions in the early stage shift to the left.

Figure 6 shows the predictions and the experimental results of the relation between laminate stress and strain. Prediction for the $[\pm 45/90]_s$ laminate overestimates the laminate stress. The discrepancy is due to occurrence of delamination and nonlinear relation between shear stress and strain.

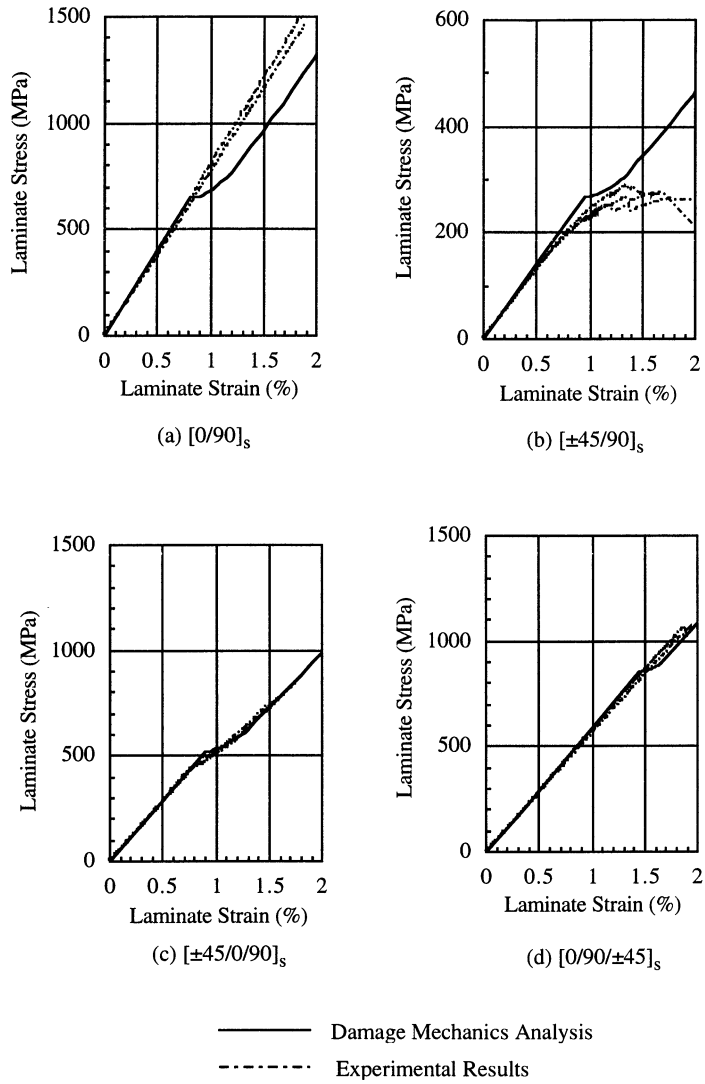


**Figure 6.** Relation between laminate stress and strain (experimental results and analytical prediction).



Prediction for the $[0/90]_s$ laminate provides larger nonlinearity than other laminates. In the present damage mechanics analysis, average transverse crack opening displacement is approximated by the average crack opening displacements for the same row of cracks in an infinite homogeneous transversely isotropic medium which has same properties as the ply in consideration. However, the existence of high modulus medium around the tip of the cracks causes a constraint effect which makes the crack opening displacement smaller. This means the present analysis evaluates the average crack opening displacements larger which result in the larger strain increments associated with transverse cracking, that is, nonlinearity of the relation between laminate stress and strain is overestimated in the present analysis. Some modification is necessary for improved prediction. The advantage of the present analysis is to predict the mechanical behavior for arbitrary laminate configurations once critical values are determined.

## 4. CONCLUSION

A damage mechanics analysis was used to predict transverse cracking in 90° ply in general composite laminates and the relation between laminate stress and strain. Predictions for transverse cracking behavior based on both the energy and stress criteria were made and compared with the experimental results. A reasonably good agreement was obtained for transverse cracking behavior. Predictions for the relation between laminate stress and strain somewhat overestimated the nonlinearity. An advantage of the present analysis is that it can predict the transverse crack behavior in laminates with arbitrary configurations.



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**APPENDIX 1**

The matrix $\mathbf{N}_\mathrm{I}^k$ in equation (3) is represented by the unit normal vector $n_j^k$ on the crack surface in ply $k$:

$$\mathbf{N}_\mathrm{I}^k = \begin{pmatrix} n_1^k & 0 & n_2^k \\ 0 & n_2^k & n_1^k \\ 0 & 0 & 0 \end{pmatrix}. \tag{A1}$$

The components of normal vector on crack surfaces always consist of in-plane component, i.e. $n_3^k = 0$.

The matrix $\beta^{ki}$, which relate the average crack opening displacements to the crack surface tractions, are written as

$$\beta^{ki} = 0, \quad \text{for all } k \neq i,$$

$$\beta^{kk} = \begin{pmatrix} \beta_1^k & 0 & 0 \\ 0 & \beta_2^k & 0 \\ 0 & 0 & \beta_3^k \end{pmatrix}, \tag{A2}$$

where

$$\begin{aligned} \beta_1^k &= \frac{4}{\pi}\gamma_1 \ln\left\{\cosh\left(\frac{\rho^k \pi}{2}\right)\right\} \Big/ \left(\rho^k\right)^2, \\ \beta_2^k &= \frac{\pi}{2}\gamma_2 \sum_{j=1}^{10} \frac{a_j}{(1+\rho^k)^j}, \\ \beta_3^k &= \frac{\pi}{2}\gamma_3 \sum_{j=1}^{9} \frac{b_j}{(1+\rho^k)^{j-2}}, \end{aligned} \tag{A3}$$

and $a_j$, $b_j$ is shown in Table A1.

**Table A1.**
Numerical parameters used in equations (A3)

| $j$ | $a$ | $b$ |
|---|---:|---:|
| 1 | 0.63666 | 0.63662 |
| 2 | 0.51806 | −0.08945 |
| 3 | 0.51695 | 0.15653 |
| 4 | −1.04897 | 0.13964 |
| 5 | 8.95572 | 0.16463 |
| 6 | −33.09444 | 0.06661 |
| 7 | 74.32002 | 0.54819 |
| 8 | −103.06411 | −1.07983 |
| 9 | 73.60337 | 0.45704 |
| 10 | −20.34326 | — |

**APPENDIX 2**

The expression of $\sigma_{TH}$ in equation (12) can be derived by the following procedure. The laminate strain $\boldsymbol{\varepsilon}$ of undamaged laminate under uniaxial tension $\boldsymbol{\sigma}_{TH} = [\sigma_{TH}\ 0\ 0]^T$ can be written as

$$\boldsymbol{\varepsilon} = \mathbf{S}_0\boldsymbol{\sigma}_{TH} + \boldsymbol{\alpha}\Delta T. \tag{A4}$$

In this condition, the ply stress in ply $k$ (90° ply) is

$$\begin{aligned} \boldsymbol{\sigma}^k &= (\mathbf{S}^k)^{-1}(\boldsymbol{\varepsilon} - \boldsymbol{\alpha}^k\Delta T) \\ &= (\mathbf{S}^k)^{-1}\left\{\mathbf{S}_0\boldsymbol{\sigma}_T + (\boldsymbol{\alpha}_0 - \boldsymbol{\alpha}^k)\Delta T\right\}. \end{aligned} \tag{A5}$$

The normal stress in the loading direction can be written as

$$\sigma_1^k = Q_{11}^k\left\{S_{11}^0\sigma_{TH} + \left(\alpha_1^0 - \alpha_1^k\right)\Delta T\right\} + Q_{12}^k\left\{S_{12}^0\sigma_{TH} + \left(\alpha_2^0 - \alpha_2^k\right)\Delta T\right\}, \tag{A6}$$

where $\sigma_i^k$, $S_{ij}^0$ and $\alpha_i^0$ are the components of $\boldsymbol{\sigma}^k$, $\mathbf{S}_0$ and $\boldsymbol{\alpha}_0$, respectively. By setting $\sigma_1^k = 0$, $\sigma_{TH}$ is expressed as

$$\sigma_{TH} = -\frac{Q_{11}^k(\alpha_1^0 - \alpha_1^k) + Q_{12}^k(\alpha_2^0 - \alpha_2^k)}{Q_{11}^k S_{11}^0 + Q_{12}^k S_{12}^0}\Delta T. \tag{A7}$$